

# A NOTE ON RATE OF CONVERGENCE IN PROBABILITY TO SEMICIRCULAR LAW

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ABSTRACT. In the present paper, we prove that under the assumption of the finite sixth moment for elements of a Wigner matrix, the convergence rate of its empirical spectral distribution to the Wigner semicircular law in probability is  $O(n^{-1/2})$  when the dimension  $n$  tends to infinity.

## 1. INTRODUCTION AND THE RESULT.

A Wigner matrix  $\mathbf{W}_n = n^{-1/2} (x_{ij})_{i,j=1}^n$  is defined to be a Hermitian random matrix whose entries on and above the diagonal are independent zero-mean random variables. It is an important model for depicting heavy-nuclei atoms, which begin with the seminal work of Wigner in 1955 ([15]). Details in this area can be found in [12].

There are various mathematical tools in the study of Wigner matrices in the past half century (see [1]). One of the most popular instruments is the limit theory of empirical spectral distribution (ESD). Here, for any  $n \times n$  matrix  $\mathbf{A}$  with real eigenvalues, the ESD of  $\mathbf{A}$  is defined by

$$F^{\mathbf{A}}(x) = \frac{1}{n} \sum_{i=1}^n I(\lambda_i^{\mathbf{A}} \leq x),$$

where  $\lambda_i^{\mathbf{A}}$  denotes the  $i$ -th smallest eigenvalue of  $\mathbf{A}$  and  $I(B)$  denotes the indicator function of an event  $B$ . It is proved that, under assumptions of for all  $i, j$ ,  $\mathbb{E}|x_{ij}|^2 = \sigma^2$ , the ESD  $F^{\mathbf{W}_n}(x)$  converges almost surely to a non-random distribution  $F(x)$  which has the destiny function

$$(1.1) \quad f(x) = \frac{1}{2\pi\sigma} \sqrt{4\sigma^2 - x^2}, \quad x \in [-2\sigma, 2\sigma].$$

This is also known as the Wigner semicircular law (see [15], [6]).

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The rate of convergence is important in establishing the central limit theorem for linear spectral statistics of Wigner matrices ([7, 6]). There are some partial results in this area. In [2], Bai proved that under the assumption of  $\sup_n \sup_{i,j} \mathbb{E}x_{ij}^4 < \infty$ , the rate of

$$\Delta_n = \|\mathbb{E}F^{\mathbf{W}^n} - F\| := \sup_x |F^{\mathbf{W}^n}(x) - F(x)|$$

tending to 0 is  $O(n^{-1/4})$ . Bai et al. in [4] obtained that the rate established in [2] is still valid for

$$\Delta_p = \|F^{\mathbf{W}^n} - F\| := \sup_x |F^{\mathbf{W}^n}(x) - F(x)|$$

Under a stronger condition that  $\sup_n \sup_{i,j} \mathbb{E}x_{ij}^8 < \infty$ , Bai et al. in [5] showed that  $\Delta_n = O(n^{-1/2})$  and  $\Delta_p = O_p(n^{-2/5})$  (Bai and Silverstein improve this condition up to  $\sup_n \sup_{i,j} \mathbb{E}x_{ij}^6 < \infty$  in their book [6]). Later, Götze et al. in [9] derived  $\Delta_n = O(n^{-1/2})$  as well assuming fourth moment, and  $\Delta_p = O_p(n^{-1/2})$  at the cost of the twelfth moment of the matrix entries. There are some other results with some special assumptions on the matrix entries. For which one can refer to [10, 11, 14, 8].

In this note we prove that the twelfth moment condition in [9] could be reduced to the sixth the moment assumption when getting  $\Delta_p = O_p(n^{-1/2})$ . Our main result of this paper is as follow.

**Theorem 1.1.** *Assume that*

- $\mathbb{E}x_{ij} = 0$ , for all  $1 \leq i \leq j \leq n$ ,
- $\mathbb{E}|x_{ii}^2| = \sigma^2 > 0$ ,  $\mathbb{E}|x_{ij}|^2 = 1$ , for all  $1 \leq i < j \leq n$ ,
- $\sup_n \sup_{1 \leq i < j \leq n} \mathbb{E}|x_{ii}^3|, \mathbb{E}|x_{ij}|^6 < \infty$ .

*Then we have*

$$(1.2) \quad \Delta_p := \|F^{\mathbf{W}^n} - F\| = O_p(n^{-1/2}).$$

**Remark 1.2.** *It is not clear what the exact rate and the optimal conditions are. As far as we know, the best known rate in the literature is  $O(n^{-1/2})$ .*

The rest of this paper is organized as follows. The main tool of proving the theorem is introduced in Section 2. Theorem 1.1 is proved in Section 3 and some technical lemmas are given in Section 4. Throughout this paper, constants appearing in inequalities are represented by  $C$  which are nonrandom and may take different values from one appearance to another.

## 2. THE MAIN TOOL

Our main tool to prove the theorem is a Berry-Esseen type inequality in [2].

**Lemma 2.1.** (*Bai inequality*) Let  $F$  be a distribution function and let  $G$  be a function of bounded variation satisfying  $\int |F(x) - G(x)|dx < \infty$ . Denote their Stieltjes transforms by  $s_F(z)$  and  $s_G(z)$  respectively, where  $z = u + iv \in \mathbb{C}^+$ . Then

$$\begin{aligned} \|F - G\| \leq & \frac{1}{\pi(1-\zeta)(2\rho-1)} \left( \int_{-A}^A |s_F(z) - s_G(z)|du \right. \\ & + 2\pi v^{-1} \int_{|x|>B} |F(x) - G(x)|dx \\ & \left. , + v^{-1} \sup_x \int_{|u|\leq 2v\epsilon} |G(x+u) - G(x)|du \right), \end{aligned}$$

where the constants  $A > B > 0$ ,  $\zeta$  and  $\epsilon$  are restricted by  $\rho = \frac{1}{\pi} \int_{|u|\leq\epsilon} \frac{1}{u^2+1} du > \frac{1}{2}$ , and  $\zeta = \frac{4B}{\pi(A-B)(2\rho-1)} \in (0, 1)$ .

Here we should notice that we can use the same methods in [9] to prove our theorem. However, Götze-Tikhomirov inequality (see Corollary 2.3 in [9]) involves the supremum of  $|s_n(z) - \mathbb{E}s_n(z)|$  over  $\Im z$  in some interval. This makes the proof rather complicated. Therefore in this paper, we use Bai inequality instead of Götze-Tikhomirov inequality which could make the presentation simpler.

### 3. THE PROOF OF THEOREM 1.1.

We will firstly introduce a new technique which can handle the moment conditions efficiently. That is given in Lemma 3.2. Then, by using this lemma and dividing the expression of  $\mathbb{E}|s_n - \mathbb{E}s_n|^2$ , we prove our theorem step by step.

Before proving the theorem, we introduce some notation. Denote  $\mathbf{I}_n$  be the identity matrix of size  $n$  and  $\mathbf{a}_i$  be the  $i$ th column of  $\mathbf{W}_n$  with  $x_{ii}$  removed. Define  $\mathbf{D}(z) = n^{-1/2}\mathbf{W}_n - z\mathbf{I}_n$ ,  $\mathbf{D}_i(z) = \mathbf{D}(z) - n^{-1}\mathbf{a}_i\mathbf{a}_i^*$  and  $s_n = s_n(z) = s_{F\mathbf{W}_n}(z)$ . Moreover write

$$\begin{aligned} \beta_i &= (n^{-1/2}x_{ii} - z - n^{-1}\mathbf{a}_i^*\mathbf{D}_i^{-1}\mathbf{a}_i)^{-1}, \quad \gamma_i = \mathbf{a}_i^*\mathbf{D}_i^{-1}\mathbf{a}_i - \text{tr}\mathbf{D}_i^{-1} \\ \varepsilon_i &= n^{-1/2}x_{ii} - n^{-1}\mathbf{a}_i^*\mathbf{D}_i^{-1}\mathbf{a}_i + \mathbb{E}s_n(z), \quad \hat{\gamma}_i = \mathbf{a}_i^*\mathbf{D}_i^{-2}\mathbf{a}_i - \text{tr}\mathbf{D}_i^{-2} \\ \xi_i &= \text{tr}\mathbf{D}^{-1} - \text{tr}\mathbf{D}_i^{-1}, \quad a_n = (z + \mathbb{E}s_n(z))^{-1}, \quad b_n = (z + 2\mathbb{E}s_n(z))^{-1} \end{aligned}$$

Throughout this section, we denote  $z = u + iv$ ,  $u \in [-16, 16]$  and  $1 \geq v \geq v_0 = C_0 n^{-1/2}$  with an appropriate constant  $C_0$ . Let  $s = s(z) = s_F(z)$ , we know that (see (3.2) in [2])

$$s(z) = -\frac{1}{2} \left( z - \sqrt{z^2 - 4} \right) \text{ for all } z \in \mathbb{C}^+.$$

Then we have

$$(3.1) \quad \int_{-16}^{16} \frac{1}{|z + 2s(z)|} du \leq \int_{-16}^{16} \frac{1}{\sqrt{|z^2 - 4|}} du \leq \int_{-16}^{16} \frac{1}{\sqrt{|u^2 - 4|}} du < 10.$$

In addition, by Lemma 2.1 and Theorem 8.2 in [6], we have for some positive constant  $C$ ,

$$(3.2) \quad \mathbb{E} \|F^{\mathbf{W}_n} - F\| \leq C \int_{-16}^{16} \mathbb{E} |s_n(z) - \mathbb{E}s_n(z)| du + O(n^{-1/2}).$$

Therefore, the rest of the proof is reduced to the lemma below.

**Lemma 3.1.** *Under the assumptions in Theorem 1.1, for any  $1 > v \geq v_0 = C_0 n^{-1/2}$  with sufficiently large  $C_0 > 0$ , we have*

$$\mathbb{E} |s_n(z) - \mathbb{E}s_n(z)|^2 \leq \frac{C}{n|z + 2s(z)|^2}.$$

**3.1. Known results and a preliminary lemma.** Following the same truncation, centralization and rescaling steps in [6], in this section we may assume the random variables satisfy the conditions as follows

$$|x_{ij}| \leq n^{1/4}, \quad \mathbb{E}x_{ij} = 0, \quad \mathbb{E}|x_{ij}|^2 = 1 \text{ for all } i, j.$$

Bai in [2] derived

$$(3.3) \quad s_n(z) = \frac{1}{n} \text{tr} \mathbf{D}^{-1} = \frac{1}{n} \sum_{i=1}^n \beta_i = -a_n + \frac{a_n}{n} \sum_{i=1}^n \beta_i \varepsilon_i.$$

For each  $i$  we have

$$|\Im \beta_i^{-1}| = |\Im (z + n^{-1} \mathbf{a}_i^* \mathbf{D}_i^{-1} \mathbf{a}_i)| \geq v,$$

which implies

$$(3.4) \quad |\beta_i| \leq v^{-1}.$$

From the definition of  $\varepsilon_i$  it follows that

$$(3.5) \quad \varepsilon_i = n^{-1/2} x_{ii} - n^{-1} \gamma_i + n^{-1} \xi_i - (s_n - \mathbb{E}s_n),$$

and

$$(3.6) \quad s_n = -a_n + \frac{a_n}{n^{3/2}} \sum_{i=1}^n \beta_i x_{ii} + \frac{a_n}{n^2} \sum_{i=1}^n \beta_i \gamma_i + \frac{a_n}{n^2} \sum_{i=1}^n \beta_i \xi_i - a_n (s_n - \mathbb{E}s_n) s_n.$$

Then, we have the the following lemma.

**Lemma 3.2.** *Under the assumption in Theorem 1.1, we have*

$$(3.7) \quad \mathbb{P}(|\beta_i| > 2) \leq \frac{C}{n^2 v^2}.$$

*Proof.* From integration by parts and Theorem 1.1 in [9], we have for  $1 > v > v_0$ ,

$$\begin{aligned} |\mathbb{E}s_n(z) - s(z)| &= \left| \int_{-\infty}^{\infty} \frac{d(\mathbb{E}F^{\mathbf{W}_n}(x) - F(x))}{x - z} \right| \\ &= \left| \int_{-\infty}^{\infty} \frac{\mathbb{E}F^{\mathbf{W}_n}(x) - F(x)}{(x - z)^2} dx \right| \leq C, \end{aligned}$$

which together with the fact that  $|s(z)| \leq 1$  ( see (3.3) in [2]) implies

$$\mathbb{E}|s_n(z)| \leq C.$$

Then, from Lemma 4.1, Lemma 4.2 and Lemma 4.3, we can check that

$$\begin{aligned} \mathbb{E}|\gamma_i|^4 &\leq C\mathbb{E}\left(\left(\text{tr}\mathbf{D}_i^{-1}(\mathbf{D}_i^{-1})^*\right)^2 + n^{1/2}\text{tr}\left(\mathbf{D}_i^{-1}(\mathbf{D}_i^{-1})^*\right)^2\right) \\ &\leq C\left(v^{-2}\mathbb{E}|\text{tr}\mathbf{D}_i^{-1}|^2 + n^{1/2}v^{-3}\mathbb{E}|\text{tr}\mathbf{D}_i^{-1}|\right) \\ (3.8) \quad &\leq \frac{Cn^2}{v^2}. \end{aligned}$$

Thus, from (3.5), Lemma 4.2 and Lemma 4.3 we have for  $v > v_0$ ,

$$(3.9) \quad \mathbb{E}|\varepsilon_i|^4 \leq \frac{C}{n^2v^2}.$$

In addition, from (8.1.19) in [6], we know that

$$(3.10) \quad |a_n| < 1 \text{ for all } z \in \mathbb{C}^+.$$

Therefore we obtain

$$\mathbb{P}(|\beta_i| > 2) \leq \mathbb{P}\left(|a_n\varepsilon_i| > \frac{1}{2}\right) \leq 2^4\mathbb{E}|\varepsilon_i|^4 \leq \frac{C}{n^2v^2}.$$

□

**3.2. The proof of Lemma 3.1.** Notice that in this subsection, we will use the equality  $\beta_i = -a_n + a_n\beta_i\varepsilon_i$  frequently. From (3.6), we have

$$\begin{aligned} \mathbb{E}|s_n - \mathbb{E}s_n|^2 &= \mathbb{E}(\overline{s_n - \mathbb{E}s_n})(s_n - \mathbb{E}s_n) \\ &= \mathbb{E}(\overline{s_n - s_n})s_n = a_n(S_1 + S_2 + S_3 + S_4) \end{aligned}$$

where

$$\begin{aligned}
S_1 &= \frac{1}{n^{3/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \\
S_2 &= -\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \gamma_i \beta_i \\
S_3 &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \xi_i \beta_i \\
S_4 &= -\mathbb{E}|s_n - \mathbb{E}s_n|^2 s_n.
\end{aligned}$$

We first consider  $S_1$ . From (3.3), we have

$$\begin{aligned}
S_1 &= \frac{1}{n^{3/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \\
&= -\frac{a_n}{n^{3/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} + \frac{a_n}{n^{3/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \varepsilon_i \\
&= S_{11} + S_{12}.
\end{aligned}$$

By (3.10) and Lemma 4.2 we have

$$(3.11) \quad |S_{11}| = \left| \frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E} \xi_i x_{ii} \right| \leq \left| \frac{a_n}{n^{5/2} v} \sum_{i=1}^n \mathbb{E} |x_{ii}| \right| \leq \frac{1}{n^{3/2} v}.$$

Applying (3.5), we obtain

$$\begin{aligned}
S_{12} &= \frac{a_n}{n^{3/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \varepsilon_i \\
&= S_{121} + S_{122} + S_{123} + S_{124},
\end{aligned}$$

where

$$\begin{aligned}
S_{121} &= \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii}^2 \beta_i \\
S_{122} &= -\frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \gamma_i \\
S_{123} &= \frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \xi_i \\
S_{124} &= -\frac{a_n}{n^{3/2}} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 x_{ii} \beta_i.
\end{aligned}$$

Using Lemma 3.2, Lemma 4.3, (3.4) and Hölder's inequality, we get

$$\begin{aligned}
|S_{121}| &= \left| \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii}^2 \beta_i \right| \\
&\leq \frac{C}{n^2} \sum_{i=1}^n (\mathbb{E} |(\overline{s_n - \mathbb{E}s_n}) x_{ii}^2| + v^{-1} \mathbb{E} |(\overline{s_n - \mathbb{E}s_n}) x_{ii}^2 I(|\beta_i| > 2)|) \\
&\leq \frac{C}{n^2} \sum_{i=1}^n (\mathbb{E} |(\overline{s_n - \mathbb{E}s_n}) x_{ii}^2|) \\
(3.12) \quad &\leq \frac{C}{n^2} \sum_{i=1}^n (\mathbb{E} |s_n - \mathbb{E}s_n|^3)^{1/3} (\mathbb{E} |x_{ii}^3|)^{2/3} = O\left(\frac{1}{n^2 v^{3/2}}\right).
\end{aligned}$$

Similarly, since  $x_{ii}$  and  $\gamma_i$  are independent, then by Lemma 4.1 and Hölder's inequality, we have

$$\begin{aligned}
|S_{122}| &= \left| \frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \gamma_i \right| \\
&\leq \frac{C}{n^{5/2}} \sum_{i=1}^n (\mathbb{E} |(s_n - \mathbb{E}s_n) x_{ii} \gamma_i|) \\
&\leq \frac{C}{n^{5/2}} \sum_{i=1}^n (\mathbb{E} |s_n - \mathbb{E}s_n|^2 \mathbb{E} |\gamma_i|^2)^{1/2} \\
(3.13) \quad &= O\left(\frac{1}{n^2 v^2}\right).
\end{aligned}$$

Using Lemma 3.2 and Lemma 4.2 again,

$$\begin{aligned}
|S_{123}| &= \left| \frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \beta_i \xi_i \right| \\
(3.14) \quad &\leq \frac{C}{n^{5/2} v} \sum_{i=1}^n (\mathbb{E} |(s_n - \mathbb{E}s_n) x_{ii}|) \leq \frac{C}{n^{5/2} v^{5/2}},
\end{aligned}$$

and

$$\begin{aligned}
|S_{124}| &= \left| \frac{a_n}{n^{3/2}} \sum_{i=1}^n \mathbb{E} |s_n - \mathbb{E}s_n|^2 x_{ii} \beta_i \right| \\
(3.15) \quad &\leq \frac{C}{n^{3/2}} \sum_{i=1}^n \mathbb{E} |s_n - \mathbb{E}s_n|^2 |x_{ii}| \leq \frac{C}{n^{5/2} v^3}.
\end{aligned}$$

Therefore combining inequalities (3.11)-(3.15) we obtain

$$(3.16) \quad |S_1| = O\left(\frac{1}{n}\right).$$

Furthermore, we have the following expression for  $S_2$ ,

$$\begin{aligned} S_2 &= -\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \gamma_i \beta_i \\ &= \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \gamma_i - \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \gamma_i \varepsilon_i \beta_i \\ &= S_{21} + S_{22} + S_{23} + S_{24} + S_{25}, \end{aligned}$$

where

$$\begin{aligned} S_{21} &= \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}(\overline{s_n - n^{-1} \text{tr} \mathbf{D}_i}) \gamma_i \\ S_{22} &= -\frac{a_n}{n^{5/2}} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) x_{ii} \gamma_i \beta_i \\ S_{23} &= \frac{a_n}{n^3} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \beta_i \gamma_i^2 \\ S_{24} &= -\frac{a_n}{n^3} \sum_{i=1}^n \mathbb{E}(\overline{s_n - \mathbb{E}s_n}) \gamma_i \beta_i \xi_i \\ S_{25} &= \frac{a_n}{n^2} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 \gamma_i \beta_i. \end{aligned}$$

Here we use the method which we used to handle the bound of  $S_1$ . Firstly, we express  $S_{21}$  as follows

$$\begin{aligned} S_{21} &= \frac{a_n}{n^3} \sum_{i=1}^n \mathbb{E}(\overline{(1 + n^{-1} \mathbf{a}_i^* \mathbf{D}_i^{-2} \mathbf{a}_i) \beta_i}) \gamma_i \\ &= S_{211} + S_{212}, \end{aligned}$$

where

$$\begin{aligned} S_{211} &= -\frac{|a_n|^2}{n^4} \sum_{i=1}^n \mathbb{E}(\overline{\hat{\gamma}_i}) \gamma_i \\ S_{212} &= \frac{|a_n|^2}{n^3} \sum_{i=1}^n \mathbb{E}(\overline{(1 + n^{-1} \mathbf{a}_i^* \mathbf{D}_i^{-2} \mathbf{a}_i) \beta_i \varepsilon_i}) \gamma_i. \end{aligned}$$



From Lemma 4.1 and Hölder's inequality we get

$$|S_{211}| \leq \frac{C}{n^4} \sum_{i=1}^n (\mathbb{E}|\hat{\gamma}_i|^2 \mathbb{E}|\gamma_i|^2)^{1/2} \leq \frac{C}{n^2 v^2}.$$

Applying Lemma 4.2, Hölder's inequality and (3.9), we obtain

$$S_{212} = \frac{|a_n|^2}{n^2} \left| \sum_{i=1}^n \mathbb{E}(\overline{s_n - n^{-1} \text{tr} \mathbf{D}_i^{-1} \varepsilon_i}) \gamma_i \right| \leq \frac{C}{n^2 v} \sum_{i=1}^n (\mathbb{E}|\varepsilon_i|^2) \mathbb{E}|\gamma_i|^2)^{1/2} \leq \frac{C}{n^2 v^2}.$$

Note that  $|S_{22}| = |S_{122}| = O(n^{-2} v^{-2})$ . And using Lemma 3.2, (3.8) and Hölder's inequality, we have

$$|S_{23}| \leq \frac{C}{n^3} \sum_{i=1}^n (\mathbb{E}|\overline{s_n - \mathbb{E}s_n} \gamma_i|^2) \leq \frac{C}{nv} (\mathbb{E}|s_n - \mathbb{E}s_n|^2)^{1/2},$$

and

$$|S_{24}| \leq \frac{C}{n^3 v} \sum_{i=1}^n (\mathbb{E}|s_n - \mathbb{E}s_n|^2) \mathbb{E}|\gamma_i|^2)^{1/2} \leq \frac{C}{n^{5/2} v^{5/2}}.$$

Now consider  $S_{25}$ , using Lemma 3.2, Lemma 4.3, Hölder's inequality and (3.9), we write

$$\begin{aligned} |S_{25}| &= \frac{|a_n|}{n^4} \left| \sum_{i=1}^n \mathbb{E}|\text{tr} \mathbf{D}_i^{-1} - \mathbb{E} \text{tr} \mathbf{D}_i^{-1}|^2 \gamma_i \beta_i \right| + O\left(\frac{1}{n^{5/2} v^{5/2}}\right) \\ &= \frac{|a_n|^2}{n^4} \left| \sum_{i=1}^n \mathbb{E}|\text{tr} \mathbf{D}_i^{-1} - \mathbb{E} \text{tr} \mathbf{D}_i^{-1}|^2 \gamma_i \varepsilon_i \beta_i \right| + O\left(\frac{1}{n^{5/2} v^{5/2}}\right) \\ &\leq \frac{C}{n^4} \sum_{i=1}^n \left( \mathbb{E}|\text{tr} \mathbf{D}_i^{-1} - \mathbb{E} \text{tr} \mathbf{D}_i^{-1}|^4 \mathbb{E}|\varepsilon_i|^4 (\mathbb{E}|\gamma_i|^2)^2 \right)^{1/4} + O\left(\frac{1}{n^{5/2} v^{5/2}}\right) \\ &= O\left(\frac{1}{n^2 v^2}\right). \end{aligned}$$

Then, we conclude that

$$(3.17) \quad |S_2| \leq \frac{C}{nv} (\mathbb{E}|s_n - \mathbb{E}s_n|^2)^{1/2} + \frac{C}{n^2 v^2}.$$

From Lemma 3.2, Lemma 4.2 and Hölder's inequality, it is easy to check that

$$(3.18) \quad |S_3| \leq \frac{C}{nv} (\mathbb{E}|s_n - \mathbb{E}s_n|^2)^{1/2}.$$

Therefore, it remains to get the bound of  $S_4$ . Now we recall the equality (3.6),

$$\begin{aligned}
S_4 &= -\mathbb{E}|s_n - \mathbb{E}s_n|^2 s_n \\
&= -\mathbb{E}s_n \mathbb{E}|s_n - \mathbb{E}s_n|^2 - \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n) \\
&= -\mathbb{E}s_n \mathbb{E}|s_n - \mathbb{E}s_n|^2 + (a_n + \mathbb{E}s_n) \mathbb{E}|s_n - \mathbb{E}s_n|^2 - \frac{a_n}{n} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 \beta_i \varepsilon_i \\
&= -\mathbb{E}s_n \mathbb{E}|s_n - \mathbb{E}s_n|^2 + (a_n + \mathbb{E}s_n) \mathbb{E}|s_n - \mathbb{E}s_n|^2 - a_n (S_{41} + S_{42} + S_{43} + S_{44}),
\end{aligned}$$

where

$$\begin{aligned}
S_{41} &= \frac{1}{n^{3/2}} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 x_{ii} \beta_i \\
S_{42} &= -\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 \gamma_i \beta_i \\
S_{43} &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}|s_n - \mathbb{E}s_n|^2 \xi_i \beta_i \\
S_{44} &= -\mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n) s_n \\
&= -\mathbb{E}s_n \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n) \\
&\quad - \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n)^2.
\end{aligned}$$

Comparing  $S_4$  with  $S_{44}$ , we obtain that

$$\begin{aligned}
&(1 + a_n \mathbb{E}s_n) \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n) \\
&= -(a_n + \mathbb{E}s_n) \mathbb{E}|s_n - \mathbb{E}s_n|^2 \\
&\quad + a_n (S_{41} + S_{42} + S_{43} - \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n)^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
&-\mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n) \\
&= b_n a_n^{-1} (a_n + \mathbb{E}s_n) \mathbb{E}|s_n - \mathbb{E}s_n|^2 \\
&\quad - b_n (S_{41} + S_{42} + S_{43} - \mathbb{E}|s_n - \mathbb{E}s_n|^2 (s_n - \mathbb{E}s_n)^2)
\end{aligned}$$

Thus denote  $\delta_n = n^{-1} \sum_{i=1}^n \mathbb{E} \beta_i \varepsilon_i$ , we conclude that

$$\begin{aligned}
S_4 &= (-\mathbb{E} s_n + b_n a_n^{-1} (a_n + \mathbb{E} s_n)) \mathbb{E} |s_n - \mathbb{E} s_n|^2 \\
&\quad - b_n (S_{41} + S_{42} + S_{43} - \mathbb{E} |s_n - \mathbb{E} s_n|^2 (s_n - \mathbb{E} s_n)^2) \\
&= (a_n - \delta_n b_n \mathbb{E} s_n) \mathbb{E} |s_n - \mathbb{E} s_n|^2 \\
&\quad - b_n (S_{41} + S_{42} + S_{43} - \mathbb{E} |s_n - \mathbb{E} s_n|^2 (s_n - \mathbb{E} s_n)^2) \\
&= (a_n + a_n \delta_n b_n) \mathbb{E} |s_n - \mathbb{E} s_n|^2 \\
&\quad - b_n (\delta_n^2 \mathbb{E} |s_n - \mathbb{E} s_n|^2 + S_{41} + S_{42} + S_{43} - \mathbb{E} |s_n - \mathbb{E} s_n|^2 (s_n - \mathbb{E} s_n)^2).
\end{aligned}$$

It is obvious that  $S_{41}$  and  $S_{124}$  have the same bound,  $S_{42}$  and  $S_{25}$  have the same bound. Using Lemma 4.2 and Lemma 4.3 we get

$$|\mathbb{E} |s_n - \mathbb{E} s_n|^2 (s_n - \mathbb{E} s_n)^2| \leq \mathbb{E} |s_n - \mathbb{E} s_n|^4 \leq \frac{C}{n^4 v^6},$$

and

$$|S_{43}| \leq \frac{1}{nv} (\mathbb{E} |s_n - \mathbb{E} s_n|^4)^{1/2} \leq \frac{C}{n^3 v^4}.$$

Furthermore, from the definition of  $\delta_i$  and (3.9), we have

$$|\delta_n| = \left| n^{-1} \sum_{i=1}^n (\mathbb{E} n^{-1} \mathbf{D}_i^{-1} - \mathbb{E} s_n + \mathbb{E} \beta_i \varepsilon_i^2) \right| \leq \frac{C}{nv}.$$

Therefore, we obtain

$$S_4 = a_n \mathbb{E} |s_n - \mathbb{E} s_n|^2 + O\left(\frac{|b_n|}{n^2 v^2}\right),$$

which combined with (3.16), (3.17) and (3.18) implies

$$|1 - a_n^2| \mathbb{E} |s_n - \mathbb{E} s_n|^2 \leq \frac{C_1 |a_n b_n|}{n} + \frac{C_2 |a_n|}{\sqrt{n}} (\mathbb{E} |s_n - \mathbb{E} s_n|^2)^{1/2}.$$

Then, from (6.91) and (6.95) in [9] which are under existing fourth moment assumption, for  $1 > v > v_0$ ,

$$|1 - a_n^2| \geq |a_n(z + 2s(z))| \text{ and } |b_n| \leq 2|z + 2s(z)|^{-1},$$

we obtain the following inequality

$$\mathbb{E} |s_n - \mathbb{E} s_n|^2 \leq \frac{C_1}{n|z + 2s(z)|^2} + \frac{C_2}{\sqrt{n}|z + 2s(z)|} (\mathbb{E} |s_n - \mathbb{E} s_n|^2)^{1/2}.$$

Solving this inequality, we obtain

$$\mathbb{E} |s_n - \mathbb{E} s_n|^2 \leq \frac{C}{n|z + 2s(z)|^2},$$

which complete the proof of the Lemma.

## 4. BASIC LEMMAS

In this section we list some results which are needed in the proof.

**Lemma 4.1.** (*Lemma B.26 of [6]*) Let  $\mathbf{A}$  be an  $n \times n$  nonrandom matrix and  $\mathbf{X} = (x_1, \dots, x_n)^*$  be a random vector of independent entries. Assume that  $\mathbb{E}x_i = 0$ ,  $\mathbb{E}|x_i|^2 = 1$ , and  $E|x_j|^l \leq \nu_l$ . Then, for any  $p \geq 1$ ,

$$\mathbb{E}|\mathbf{X}^* \mathbf{A} \mathbf{X} - \text{tr} \mathbf{A}|^p \leq C_p \left( (\nu_4 \text{tr}(\mathbf{A} \mathbf{A}^*))^{p/2} + \nu_{2p} \text{tr}(\mathbf{A} \mathbf{A}^*)^{p/2} \right),$$

where  $C_p$  is a constant depending on  $p$  only.

**Lemma 4.2.** (*Lemma 2.6 of [13]*). Let  $z \in \mathbb{C}^+$  with  $v = \Im z$ ,  $\mathbf{A}$  and  $\mathbf{B}$   $n \times n$  with  $\mathbf{B}$  Hermitian,  $\tau \in \mathbb{R}$ , and  $\mathbf{q} \in \mathbb{C}^N$ . Then

$$|\text{tr}((\mathbf{B} - z\mathbf{I})^{-1} - (\mathbf{B} + \tau \mathbf{q} \mathbf{q}^* - z\mathbf{I})^{-1}) \mathbf{A}| \leq \frac{\|\mathbf{A}\|}{v}.$$

**Lemma 4.3.** (*Lemma 8.7 of [6]*) Under the assumption in Theorem 1.1, we have

$$(4.1) \quad \mathbb{E}|s_n(z) - \mathbb{E}s_n(z)|^{2l} \leq \frac{C}{n^{2l} v^{3l}}.$$

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